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Well-posedness without semicontinuity for parametric quasiequilibria and quasioptimization☆

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ABSTRACT

We consider quasiequilibrium and quasioptimization problems. A relaxed level closedness notion is proposed and used together with pseudocontinuity to establish sufficient conditions for parametric well-posedness and well-posedness without semicontinuity assumptions. We prove them in general formulations, though such relaxations allow us to improve some existing results even in simple cases of R^1. Several new well-posedness results are also obtained. For topological settings we use sensitivity analysis while for problems on metric spaces we argue on diameters and Kuratowski's and Hausdorff's measures of noncompactness of approximate solution sets.

1. Introduction

In their seminal papers, Hadamard [1] and Tikhonov [2] initiated two ways of developing a well-posedness study for various mathematical problems. For constrained optimization the pioneer work was [3] of Levitin and Polyak, who extended the definition for unconstrained problems in [2]. Observe that the notions of Hadamard and Tikhonov were proved closely related in [4,5]. Recently, these two notions have been more blended and linked to stability theory in parametric well-posedness study [6–12]. Well-posedness for various problems related to optimization has been recently intensively considered, see e.g.: for optimization problems [5,9,12–16], for variational inequalities [17–21], for Nash equilibria [22,23], for fixed-point problems [8,19,24], for inclusion problems [8,19,24] and for equilibrium problems [6,7,25]. In most cases it is commonly assumed at least that the involved functions are lower semicontinuous. But in many practical optimization and control problems we meet even nonsemicontinuous functions. In [9,26] a weaker notion of lower pseudocontinuity is introduced to investigate parametric constrained optimization. In this paper we propose generalized level closedness definitions and use them together with pseudocontinuity to consider well-posedness in the Tikhonov sense, which is more important in approximation study and numerical algorithms, because all algorithms consist of providing sequences of approximate solutions convergent to an exact one. Simple examples (e.g. Examples 2.1 and 2.2) ensure that these properties are properly weaker than semicontinuity and hence results under assumptions about these properties are significant in practical situations. Note that quasiequilibrium models contain quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, etc. A quasioptimization problem is more general than an optimization one as constraint sets depend on the decision

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($f(x_0) \leq \liminf f(x_n)$, resp.). Note that in this paper we are concerned always with sequential properties. Hence we write clearly "sequential" or "sequentially" only to remind the reader in case necessary. Observe that f is usc at x_0 if and only if for all $\{x_n\} \to x_0$ and all $b \in R$,

$$[f(x_n) \geq b, \ \forall n] \Rightarrow [f(x_0) \geq b]$$

and similarly for lower semicontinuity. Therefore, we propose the following natural definition.

Definition 2.1. Let X and Y be topological spaces, $f : X \to \bar{R}$ and $g : Y \to \bar{R}$.

(a) f is called (sequentially) upper 0-level closed with respect to (w.r.t.) g at $(x_0, y_0) \in X \times Y$ if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0),

$$[f(x_n) + g(y_n) \geq 0, \ \forall n] \Rightarrow [f(x_0) + g(y_0) \geq 0].$$

(b) f is called (sequentially) lower 0-level closed w.r.t. g at (x_0, y_0) if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0),

$$[f(x_n) + g(y_n) \leq 0, \ \forall n] \Rightarrow [f(x_0) + g(y_0) \leq 0].$$

If we have f in place of $f + g$ in the above inequalities, we say that f is upper (or lower) 0-level closed at x_0. While if we have $b \in R$ instead of 0, then of course "0-level" is replaced by "b-level".

Remark 2.1. If f and g are usc (lsc, resp.) at x_0 and y_0, respectively, then f is upper (lower, resp.) 0-level closed w.r.t. g at (x_0, y_0). Indeed, if $\{(x_n, y_n)\} \to (x_0, y_0)$ and $f(x_n) + g(y_n) \geq 0$ for all n, one has

$$f(x_0) + g(y_0) \geq \limsup f(x_n) + \limsup g(y_n) \geq \limsup [f(x_n) + g(y_n)] \geq 0.$$

From now on we use id to denote the identity map on R_+. The following example shows that the converse of the above remark is not true.

Example 2.1. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in Q, \\ 1, & \text{if } x \in R \setminus Q, \end{cases}$$

where Q is the set of the rational numbers. Then f is upper 0-level closed w.r.t. id at (x, y), for all $(x, y) \in R \times R_+$, but f is neither usc at any $x \in Q$ nor lsc at any $x \in R \setminus Q$.

Definition 2.2 ([9,26]). Let X be a topological space and $f : X \to \bar{R}$.

(a) f is said to be (sequentially) upper pseudocontinuous at $x_0 \in X$ if,

$$[f(x) > f(x_0)] \Rightarrow [\text{for any } \{x_n\} \to x_0, f(x) > \limsup f(x_n)].$$

(b) f is called lower pseudocontinuous at $x_0 \in X$ if,

$$[f(x) < f(x_0)] \Rightarrow [\text{for any } \{x_n\} \to x_0, f(x) < \liminf f(x_n)].$$

(c) f is termed pseudocontinuous at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

The class of the upper pseudocontinuous functions strictly contains that of the usc functions, see [26]. We include here a new simple illustrative example.

Example 2.2. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ x - 1, & \text{if } x < 0. \end{cases}$$

Then, f is pseudocontinuous at 0 but neither usc nor lsc at 0.

We note further that if f and g are lsc (or usc) at x_0 then $f + g$ is lsc (usc, resp.) at x_0. Unfortunately, this property does not hold for pseudocontinuous functions as shown by

Example 2.3. Let $f_1, g_1 : R \to R$ be defined as follows

$$f_1(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ \dfrac{x}{2}, & \text{if } x < 0 \end{cases} \quad \text{and} \quad g_1(x) = -x.$$

The assumptions of Theorem 3.1 are essential as indicated in the following examples.

Example 3.1 (*The Compactness of X Cannot be Dropped*). Let $X = R$, $\Lambda = R_+$, $K_1(x,\lambda) = K_2(x,\lambda) = R$, $\bar{\lambda} = 0$ and $f(x,y,\lambda) = 2^{x-y} + \lambda$. It is clear that in $X \times \Lambda$, K_1 is closed and K_2 is lsc. (ii) holds as f is continuous in $X \times X \times \Lambda$. But $S(\lambda) = R$ for all $\lambda \in \Lambda$. Hence, (QEP) is not well-posed at 0. Indeed, let $\lambda_n = \frac{1}{n} \to 0$ and $x_n = n \in S(\bar{\lambda}_n)$ for all n. It is clear that $\{x_n\}$ has no convergent subsequence. The reason is that X is not compact.

Example 3.2 (*The Closedness of K_1 is Essential*). Let $X = [-2, 1]$, $\Lambda = [0, 1]$, $K_1(x,\lambda) = (-2\lambda, 1]$, $K_2(x,\lambda) = [0, 1]$, $\bar{\lambda} = 0$ and $f(x,y,\lambda) = x(x-y)$. It is not hard to see that X is compact, K_2 is lsc in $X \times \Lambda$, (ii) is fulfilled (by the continuity of f). But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Therefore, (QEP) is not well-posed at 0. The reason is that K_1 is not closed at $X \times \{0\}$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = -\frac{1}{n} \in K_1(x_n, \lambda_n) = (-\frac{2}{n}, 1]$. We see that $\{z_n\}$ tends to $0 \notin K_1(0, 0)$.

Example 3.3 (*The Lower Semicontinuity of K_2 Cannot be Dispensed*). Let $X = [-1, 1]$, $\Lambda = [0, 1]$. $K_1(x,\lambda) = [0, 1]$, $f(x,y,\lambda) = x + y$, $\bar{\lambda} = 0$ and

$$K_2(x,\lambda) = \begin{cases} \{-1, 0, 1\}, & \text{if } \lambda = 0, \\ \{0, 1\}, & \text{otherwise.} \end{cases}$$

Then X is compact, K_1 is closed in $X \times \Lambda$ and (ii) holds (by the continuity of f in $X \times X \times \Lambda$). But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Thus, (QEP) is not well-posed at 0. The reason is that K_2 is not lsc in $X \times \{\bar{\lambda}\}$.

Example 3.4 (*(ii) Cannot be Dropped*). Let $X = [0, 1]$, $\Lambda = [0, 1]$, $K_1(x,\lambda) \equiv K_2(x,\lambda) = [0, 1]$ and

$$f(x,y,\lambda) = \begin{cases} x - y, & \text{if } \lambda = 0, \\ y - x, & \text{otherwise.} \end{cases}$$

It is clear that assumption (i) is satisfied and $S(0) = \{1\}$. Let $\lambda_n = \varepsilon_n = \frac{1}{n}$, and $x_n = 0 \in \widetilde{S}(\lambda_n, \varepsilon_n)$. Then $\{x_n\}$ is an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$. But $\{x_n\} \to 0 \notin S(0)$ and hence $\{(\mathrm{QEP}_\lambda) : \lambda \in \Lambda\}$ is not well-posed at $\bar{\lambda} = 0$. The reason is that assumption (ii) is violated. Indeed, taking $x_n = 0$, $y_n = 1$, $\lambda_n = \frac{1}{n}$ and $\varepsilon_n = 0$, we have $\{(x_n, y_n, \lambda_n, \varepsilon_n)\} \to (0, 1, 0, 0)$ and $f(x_n, y_n, \lambda_n) + \varepsilon_n = f(0, 1, \frac{1}{n}) = 1 > 0$ but $f(0, 1, 0) = -1 < 0$.

Remark 3.2. In the special case where $K(x,\lambda) \equiv X$, it is not hard to check that the assumption (ii) for f can be reduced to the same condition for $f(.,y,.)$, for all $y \in X$. Therefore, Theorem 3.1 improves Theorem 3.3 in [25]. Indeed, it suffices to check assumption (ii) of Theorem 3.1 from the (assumed in [25]) monotonicity of $f(.,.,\bar{\lambda})$ and lower semicontinuity of $f(x,.,.)$. If $\{(x_n, \lambda_n)\} \to (x, \bar{\lambda})$ and $\{\varepsilon_n\}$ tends to 0^+ are such that

$$f(x_n, y, \bar{\lambda}_n) + \varepsilon_n \geq 0,$$

then, by the monotonicity, the inequalities

$$f(y, x, \bar{\lambda}) \leq \liminf f(y, x_n, \lambda_n) \leq \liminf(-f(x_n, y, \lambda_n)) \leq \liminf \varepsilon_n = 0$$

imply that $f(x, y, \bar{\lambda}) \geq 0$. Note further that we omit the hemicontinuity of $f(.,.,\bar{\lambda})$ and convexity of $f(x,.,\bar{\lambda})$ imposed in [25].

Theorem 3.2. *Let X and Λ be metric spaces.*

(i) *If* (QEP) *is uniquely well-posed at $\bar{\lambda}$, then* diam $\Pi(\bar{\lambda}, \zeta, \varepsilon) \to 0^+$ *as* $(\zeta, \varepsilon) \to (0^+, 0^+)$.
(ii) *Conversely, if X is complete and the following conditions hold*
  (a) *K_1 is closed and K_2 is lsc in $X \times \{\bar{\lambda}\}$;*
  (b) *f is upper 0-level closed w.r.t.* id *in $K_1(X, \bar{\lambda}) \times K_2(X, \bar{\lambda}) \times \{\bar{\lambda}\} \times \{0\}$,*
  then (QEP) *is uniquely well-posed at $\bar{\lambda}$, provided that* diam $\Pi(\bar{\lambda}, \zeta, \varepsilon) \to 0^+$ *as* $(\zeta, \varepsilon) \to (0^+, 0^+)$.

Proof. (i) Suppose (QEP) is uniquely well-posed at $\bar{\lambda}$, but there is $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ such that there are $n_0 \in \mathcal{N}$ (the set of natural numbers) and $r > 0$ such that, for all $n \geq n_0$,

$$\operatorname{diam} \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n) > r.$$

Then, there exist $x_n^1, x_n^2 \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that $d(x_n^1, x_n^2) > \frac{r}{2}$. Consequently, there are $\lambda_n^1, \lambda_n^2 \in B(\bar{\lambda}, \zeta_n)$ such that

$$f(x_n^1, y, \lambda_n^1) + \varepsilon_n \geq 0, \quad \forall y \in K(x_n^1, \lambda_n^1)$$

and

$$f(x_n^2, y, \lambda_n^2) + \varepsilon_n \geq 0, \quad \forall y \in K(x_n^2, \lambda_n^2),$$

Proof. Let γ be the Hausdorff measure η (for the Kuratowski measure case the argument is similar).

(i) Assume that (QEP) is well-posed at $\bar{\lambda}$ and $(\zeta, \varepsilon) \to (0^+, 0^+)$. Since $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda}, \zeta, \varepsilon)$ for all $\zeta, \varepsilon > 0$,

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})),$$

where $H^*(A, B) = \sup_{a\in A} d(a, B)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Since $\{x_n\}$ is an approximating sequence for (QEP), there is a subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

If $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon)$, then

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^{n} B\left(z_k, \varepsilon + H\left(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\right)\right)$$

and hence

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H\left(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\right) + \gamma(S(\bar{\lambda}))\,.$$

Since $S(\bar{\lambda})$ is compact, $\eta(S(\bar{\lambda})) = 0$. So we have

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Now we claim that $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. Indeed, suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ and $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$, $d(x_n, S(\bar{\lambda})) \geq \rho$. Since $\{x_n\}$ is an approximating sequence for (QEP), there is a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction.

(ii) Assume that $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. We first prove that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε. Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ be such that $\{x_n\} \to x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x_n, \lambda_n)$,

$$f(x_n, y, \lambda_n) + \varepsilon \geq 0.$$

Since $B(\bar{\lambda}, \zeta)$ is compact, we can assume that $\{\lambda_n\} \to \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. By the closedness of K_1 at (x, λ), $x \in K_1(x, \lambda)$. We claim that, for all $y \in K_2(x, \lambda)$,

$$f(x, y, \lambda) + \varepsilon \geq 0.$$

Indeed, if there exists $y \in K_2(x, \lambda)$ such that $f(x, y, \lambda) + \varepsilon < 0$, there is $y_n \in K_2(x_n, \lambda_n)$ such that $\{y_n\} \to y$ as K_2 is lsc at (x, λ). By the upper $-\varepsilon$-level closedness of f at (x, y, λ), there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0, f(x_n, y_n, \lambda_n) < -\varepsilon$, a contradiction. Since $\lambda \in B(\bar{\lambda}, \zeta)$, we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed.

Now we show that $S(\bar{\lambda}) = \bigcap_{\zeta>0,\varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. We first check that $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \widetilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, it is easy to see that $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \supseteq \widetilde{S}(\bar{\lambda}, \varepsilon)$. Let $x \in \bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. There is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x, \lambda_n)$, $f(x, y, \lambda_n) + \varepsilon \geq 0$. Since $x \in K_1(x, \lambda_n)$, $\{\lambda_n\} \to \bar{\lambda}$ and K_1 is closed, one sees that $x \in K_1(x, \bar{\lambda})$. Now we verify that $x \in \widetilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, for each $y \in K_2(x, \bar{\lambda})$, since K_2 is lsc at $(x, \bar{\lambda})$, there exists $y_n \in K_2(x, \lambda_n)$ with $\{y_n\} \to y$. Since $x \in \widetilde{S}(\lambda_n, \varepsilon)$,

$$f(x, y_n, \lambda_n) + \varepsilon \geq 0.$$

By the upper $-\varepsilon$-level closedness of f, one has

$$f(x, y, \bar{\lambda}) + \varepsilon \geq 0,$$

i.e. $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \widetilde{S}(\bar{\lambda}, \varepsilon)$. Hence, $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \widetilde{S}(\bar{\lambda}, \varepsilon)$. Next, we have $S(\bar{\lambda}) = \bigcap_{\varepsilon>0} \widetilde{S}(\bar{\lambda}, \varepsilon) = \bigcap_{\zeta>0,\varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$.

Since $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$, the regular measure properties of η imply that $S(\bar{\lambda})$ is compact and $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.

Let x_n be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \to \bar{\lambda}$. There is $\{\varepsilon_n\} \to 0^+$ such that, for all $y \in K_2(x_n, \lambda_n)$ and all $n \in \mathcal{N}$,

$$f(x_n, y, \lambda_n) + \varepsilon_n \geq 0.$$

This means that $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \to 0^+.$$

Hence, there is $\bar{x}_n \in S(\bar{\lambda})$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ convergent to some point $\bar{x}$ of $S(\bar{\lambda})$. Therefore, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ tends to $\bar{x}$. Hence, (QEP) is well-posed at $\bar{\lambda}$. □

The following examples show that the assumptions of Theorem 3.3(ii) are essential.

Example 3.6 (*The Closedness of K_1 cannot be Dispensed*). Let $X = R$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = (-\lambda, 1]$, $K_2(x, \lambda) \equiv [0, 1]$, $f(x, y, \lambda) = x(x - y)$ and $\bar{\lambda} = 0$. It is easy to see that X is complete, Λ is compact, K_2 is lsc in $X \times \Lambda$. Condition (ii)(b) holds

Now suppose ad absurdum that $g(y, \bar{\lambda}) < g(x, \bar{\lambda})$. By Lemma 2.1 we have

$$\limsup g(y_n, \lambda_n) < \liminf g(x_n, \lambda_n).$$

Hence, there are $t_1, t_2 \in R$ and $n_0 \in \mathcal{N}$ such that, for $n \geq n_0$,

$$g(y_n, \lambda_n) \leq t_1 < t_2 \leq g(x_n, \lambda_n)$$

and then

$$g(y_n, \lambda_n) - g(x_n, \lambda_n) \leq t_1 - t_2 < 0,$$

which is impossible and we are done. □

Let $m : X \times \Lambda \to R$ be the following kind of marginal functions

$$m(x, \lambda) := \inf\{g(y, \lambda) \mid y \in K(x, \lambda)\}.$$

When (QOP) is given on metric spaces, similarly as for (QEP) we define $\widetilde{S}$ and Π as follows

$$\widetilde{S}(\lambda, \varepsilon) = \{x \in K(x, \lambda) \mid g(x, \lambda) \leq m(x, \lambda) + \varepsilon\},$$
$$\Pi(\bar{\lambda}, \zeta, \varepsilon) = \bigcup_{\lambda \in B(\bar{\lambda}, \zeta)} \widetilde{S}(\lambda, \varepsilon).$$

Theorem 4.2. *Assume that*

(i) *X is compact and K is closed in $X \times \{\bar{\lambda}\}$;*
(ii) *g is lower pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$;*
(iii) *m is usc in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$.*

Then (QOP) *is well-posed at $\bar{\lambda}$. Furthermore, if* (QOP) *has a unique solution, it is uniquely well-posed at $\bar{\lambda}$.*

Proof. We check first that $\widetilde{S}$ is usc at $(\bar{\lambda}, 0)$. Suppose to the contrary the existence of an open superset U of $\widetilde{S}(\bar{\lambda}, 0)$ such that for all $\{(\lambda_n, \varepsilon_n)\}$ convergent to $(\bar{\lambda}, 0^+)$ in $\Lambda \times R_+$, there is $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$ such that $x_n \notin U$, for all n. By the compactness of X one can assume that $\{x_n\}$ tends to some x_0. Since K is closed at $(x_0, \bar{\lambda})$, $x_0 \in K(x_0, \bar{\lambda})$. If $x_0 \notin \widetilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$, there is $y_0 \in K(x_0, \bar{\lambda})$ such that $g(y_0, \bar{\lambda}) < g(x_0, \bar{\lambda})$. Since g is lower pseudocontinuous at $(x_0, \bar{\lambda})$, we have

$$m(x_0, \bar{\lambda}) \leq g(y_0, \bar{\lambda}) < \liminf g(x_n, \lambda_n).$$

The upper semicontinuity of m at $(x_0, \bar{\lambda})$ yields some $t \in R$ such that

$$\limsup m(x_n, \lambda_n) < t < \liminf g(x_n, \lambda_n).$$

Hence, there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$m(x_n, \lambda_n) - g(x_n, \lambda_n) < t - g(x_n, \lambda_n).$$

As $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$,

$$-\varepsilon_n \leq m(x_n, \lambda_n) - g(x_n, \lambda_n) \leq 0.$$

Therefore,

$$0 = \lim_{n \to +\infty} [m(x_n, \lambda_n) - g(x_n \lambda_n)] \leq t - \liminf_{n \to +\infty} g(x_n, \lambda_n) < 0.$$

This contradiction shows that $x_0 \in S(\bar{\lambda})$. Then another contradiction is obtained as $x_n \notin U$. Thus, $\widetilde{S}$ is usc at $(\bar{\lambda}, 0)$. Now we prove that $S(\bar{\lambda})$ is compact by checking its closedness. Let $\{x_n\} \subseteq S(\bar{\lambda})$ converge to x_0. As $S(\bar{\lambda}) \subseteq \widetilde{S}(\bar{\lambda}, \varepsilon_n)$, by the preceding argument one sees that $x_0 \in S(\bar{\lambda})$. By Remark 3.1, (QOP) is well-posed at $\bar{\lambda}$. □

The following examples explain that Theorems 4.1 and 4.2 are incomparable and each of them may be applicable in different situations.

Example 4.1. Let $X = \Lambda = [0, 1]$, $K(x, \lambda) = [0, 1]$, $\bar{\lambda} = 1$ and

$$g(x, \lambda) = \begin{cases} (1 + x)(1 - \lambda), & \text{if } 0 \leq \lambda < 1, \\ -1, & \text{if } \lambda = 1. \end{cases}$$

It is clear that K is continuous, X is compact and g is lower pseudocontinuous in $[0, 1] \times [0, 1]$. Now we check that g is upper pseudocontinuous at $(x, 1)$, for all $x \in [0, 1]$. Indeed, assume that $g(y, \lambda) > g(x, 1) = -1$ and $\{(x_n, \lambda_n)\} \to (x, 1)$. It is clear that, $g(y, \lambda) > 0$ as $\lambda < 1$ and $\limsup_{n \to +\infty} g(x_n, \lambda_n) = 0$. So $g(y, \lambda) > \limsup_{n \to +\infty} g(x_n, \lambda_n)$. Hence, the assumptions of Theorem 4.1 are satisfied and we obtain the well-posedness at 1 (in fact, $S(1) = [0, 1]$ and $S(\lambda) = \{0\}$ for all $0 \leq \lambda < 1$).

(ii) *Conversely, assume that X is complete and Λ is compact or finite dimensional. Impose further that,*
(a) *K is closed in $X \times \Lambda$;*
(b) *g is lsc in $K(X, \Lambda) \times \Lambda$;*
(c) *m is usc in $K(X, \Lambda) \times \Lambda$.*
Then (QOP) *is well-posed at $\bar{\lambda}$, provided that $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.*

Proof. By similarity we discuss only the case where $\gamma = \mu$, the Kuratowski measure.

(i) Assume that (QOP) is well-posed at $\bar{\lambda}$. Since, for all positive ζ and ε, $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda}, \zeta, \varepsilon)$, one has

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Let $\{x_n\}$ be a sequence in $S(\bar{\lambda})$. Then $\{x_n\}$ is an approximating sequence for (QOP) and has a subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

Let $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$ with $\operatorname{diam} M_k \leq \varepsilon$, for $k = 1, \ldots, n$. Set

$$N_k = \{z \in X \mid d(z, M_k) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))\}.$$

We claim that

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^{n} N_k.$$

Indeed, let $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Then $d(x, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$. Since $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, we see that $d(x, \bigcup_{k=1}^{n} M_k) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$. Hence, there is $\bar{k}$ such that $d(x, M_{\bar{k}}) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda}))$, i.e. $x \in N_{\bar{k}}$. So, $\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Note further that

$$\operatorname{diam} N_k = \operatorname{diam} M_k + 2H\left(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\right) \leq \varepsilon + 2H\left(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\right),$$

and hence, as $\mu(S(\bar{\lambda})) = 0$,

$$\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq 2H\left(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\right) + \mu(S(\bar{\lambda})) = 2H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Now we prove that $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. Suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ and $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$, $d(x_n, S(\bar{\lambda})) \geq \rho$. Since $\{x_n\}$ is an approximating sequence for (QOP), it has a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction. Therefore, $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.

(ii) Assume that $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. We first show that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε. Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\{x_n\} \to x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon.$$

Because $B(\bar{\lambda}, \zeta)$ is compact, we assume that $\{\lambda_n\} \to \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. Since K is closed at (x, λ), $x \in K(x, \lambda)$. By the lower semicontinuity of g and the upper semicontinuity of m at (x, λ), we have

$$g(x, \lambda) \leq m(x, \lambda) + \varepsilon.$$

As $\lambda \in B(\bar{\lambda}, \zeta)$ we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed. Note further that $S(\bar{\lambda}) = \bigcap_{\zeta>0, \varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. From the properties of μ it follows that $S(\bar{\lambda})$ is compact and $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+$. Let $\{x_n\}$ be an approximating sequence for (QOP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \to \bar{\lambda}$. There is $\{\varepsilon_n\} \to 0^+$ such that, for all $n \in \mathcal{N}$,

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon_n.$$

Consequently, $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \to 0^+.$$

By the compactness of $S(\bar{\lambda})$, there is a subsequence of $\{x_n\}$ converging to some point of $S(\bar{\lambda})$. Hence, (QOP) is well-posed at $\bar{\lambda}$. $\square$

Theorem 4.5. *Assume that X is complete and Λ is compact or finite dimensional. Let the following conditions hold*

(a) *K is closed and lsc in $X \times \Lambda$;*
(b) *g is continuous in $K(X, \Lambda) \times \Lambda$.*

Then (QOP) *is well-posed at $\bar{\lambda}$, provided that $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.*

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