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A weak law of large numbers and Poisson approximation problem for extension random sums of *m*-dependent random variables

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ABSTRACT

In probability theory, Limit theorems applied in many fields are often considered for independent random variables. However, there are several studied models presented that random variables depend on each other. The aim of current paper is to study an extension of random sums of m-dependent random variables. Weak law of large numbers and Poisson approximation problem for random sums are established. The received results are extensions and generalizations of some known results.

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1 INTRODUCTION

The sequence of *m*-dependent random variables was first mentioned by Hoffding and Robbins (1948), followed by some other authors such as Dianada (1955), Van der Genugten (1989) and Shang (2012). Up to now, the structure of *m*-dependent random variables has been interested by many authors because of its applicability in economic models. For example, the *m*-dependent structures are applied on moving average and descent processes (Islak, 2016).

A sequence of random variables $\{X_i\}_{i\geq 1}$ is called *m*-dependent for a given fixed m if for any two subsets I and J of $\{1, 2,\ldots\}$ such that $\min(J)-max(I)>m$, the families $(X_i)_{i\in I}$ and $(X_j)_{j\in J}$ are independent. Equivalently, a sequence is *m*-dependent if two sets of random variables $\{X_1,X_2,\ldots,X_i\}$ and $\{X_j,X_{j+1},\ldots\}$ are independent whenever $j-i>m$ (Shang, 2012).

Let N_n (n = 1, 2, …) be a random variable that receives non-negative and integer values, and $X_1,X_2,\ldots,X_n,\ldots$ be independent, identically distributed random variables and independent of N_n. Then, the random variable

$$S_{N_n}=\begin{cases} 0, & N_n=0,\\ X_1+X_2+\ldots+X_n, & N_n=n.\end{cases} \tag{1}$$

is called a random sum.

In this paper, a more general case is considered when the random variables $X_1,X_2,\ldots$ are *m*-dependent, and the index N_n is the sum of random variables, namely $N_n=Y_1+Y_2+\ldots+Y_n$, where Y_i

$(i=1,2,...,n)$ are independent, non-negative and integer valued random variables. Then, S_{N_n} is called the extension random sums of the m-dependent random variables.

In addition, it is necessary to recall the following property for proving theorems:

$$N_n \xrightarrow{P} \infty \text{ as } n\to\infty. \quad (2)$$

2 MAIN RESULTS

2.1 The law of large numbers for the extension random sums of *m*-dependent random variables

In this section, we will solve the law of large numbers for the extension random sums of m-dependent random variables, instead of just considering for the case of independent random variables (Tran Loc Hung and Le Truong Giang, 2016).

When studying about the asymptotic shape of random sums of m-dependent variables, an m-dependent sequence and define its characteristic function need to be constructed. One of the most common ways is to build m-dependent sequences from a sequence of independent random variables by setting $X_i=f(Z_i,...,Z_{i+m})$.

If random variables $Z_1,Z_2,...$ are independent and identically distributed, the stopping and m-dependent sequence $\{X_i\}_{i\geq 1}$ can be created by choosing the appropriate function f (Shang, 2012; Islak, 2016). Let $\{Z_i\}_{i\geq 1}$ be a sequence of independent, identically distributed random variables with $E(Z_i)=\mu$. Set $X_i=\frac{Z_i+Z_{i+1}+\cdots+Z_{i+m}}{m+1}$. Then, $\{X_i\}_{i\geq 1}$ is a sequence of m-dependent, identically distributed random variables sequence with $E(X_i)=\mu$. Set

$$\begin{aligned}S_n &= \sum_{i=1}^{n} X_i \\ &= \frac{Z_1+2Z_2+\cdots+mZ_m+mZ_{n+1}+\cdots+2Z_{n+m-1}+Z_{n+m}}{m+1} \\ &+ \sum_{i=m+1}^{n} Z_i. \quad (3)\end{aligned}$$

It is easy to gain the following characteristic function of S_n.

$$\begin{aligned}\varphi_{S_n}(t) &= \varphi_{Z_1}\left(\frac{t}{m+1}\right)\varphi_{Z_2}\left(\frac{2t}{m+1}\right)\cdots\varphi_{Z_m}\left(\frac{mt}{m+1}\right) \\ &\times\varphi_{Z_{n+1}}\left(\frac{mt}{m+1}\right)\cdots\varphi_{Z_{n+m-1}}\left(\frac{2t}{m+1}\right)\varphi_{Z_{n+m}}\left(\frac{t}{m+1}\right)\left[\varphi_{Z_1}(t)\right]^{n-m} \\ &= \varphi^2_{Z_1}\left(\frac{t}{m+1}\right)\varphi^2_{Z_1}\left(\frac{2t}{m+1}\right)\cdots\varphi^2_{Z_1}\left(\frac{mt}{m+1}\right)\left[\varphi_{Z_1}(t)\right]^{n-m}. \quad (4)\end{aligned}$$

Theorem 2.1. Let $\{X_i\}_{i\geq 1}$ be the m-dependent sequence defined as above. Let $Y_1,Y_2,...,Y_n$ be independent, identically distributed, non- negative, integer-valued random variables with $E(Y_i)=\alpha$. Then,

$$\frac{S_{N_n}}{n} \xrightarrow{P} \alpha\mu, \quad as \quad n\to\infty, \quad (5)$$

where $S_{N_n}:=X_1+X_2+...+X_{N_n}$ and $N_n=Y_1+Y_2+...+Y_n$.

Proof.

It is simple to see that

$$\begin{aligned}\varphi_{\frac{S_n}{n}}(t) &= \varphi^2_{Z_1}\left(\frac{t}{n(m+1)}\right)\varphi^2_{Z_1}\left(\frac{2t}{n(m+1)}\right) \\ &\cdots\varphi^2_{Z_1}\left(\frac{mt}{n(m+1)}\right)\left[\varphi_{Z_1}\left(\frac{t}{n}\right)\right]^{n-m}.\end{aligned} \quad (6)$$

Take the logarithms of two sides, we gain

$$\begin{aligned}\ln\varphi_{\frac{S_n}{n}}(t) &= \ln\left[\begin{array}{l}\varphi^2_{Z_1}\left(\frac{t}{n(m+1)}\right)\varphi^2_{Z_1}\left(\frac{2t}{n(m+1)}\right) \\ \cdots\varphi^2_{Z_1}\left(\frac{mt}{n(m+1)}\right)\left[\varphi_{Z_1}\left(\frac{t}{n}\right)\right]^{n-m}\end{array}\right] \\ &= 2\left[\ln\varphi_{Z_1}\left(\frac{t}{n(m+1)}\right)+\cdots+\ln\varphi_{Z_1}\left(\frac{mt}{n(m+1)}\right)\right] \\ &+ (n-m)\ln\varphi_{Z_1}\left(\frac{t}{n}\right). \quad (7)\end{aligned}$$

Applying Taylor expansion of the first order, we get

$$\varphi_{Z_1}(t)=\varphi_{Z_1}(0)+\varphi'_{Z_1}(0)\frac{t}{1!}+o(t)=1+i\mu t+o(t). \quad (8)$$

Where, $o(t)$ is a convergence rate type of small-o.

Therefore,

$$\begin{aligned}\varphi_{Z_1}\left(\frac{kt}{n(m+1)}\right) &= 1+i\mu\frac{kt}{n(m+1)} \\ &+ o\left(\frac{kt}{n(m+1)}\right), k=1,2,...,m. \quad (9) \\ \varphi_{Z_1}\left(\frac{t}{n}\right) &= 1+i\mu\frac{t}{n}+o\left(\frac{t}{n}\right).\end{aligned}$$

It is easy to check that

$$\lim_{n\to\infty} \ln\varphi_{Z_1}\left(\frac{kt}{n(m+1)}\right) = \lim_{n\to\infty} \ln\left[1+i\mu\frac{kt}{n(m+1)}+o\left(\frac{kt}{n(m+1)}\right)\right]$$

$$= \lim_{n\to\infty}\frac{i\mu kt}{n(m+1)},\ k=1,2,...,m.$$

$$\lim_{n\to\infty} \ln\varphi_{Z_1}\left(\frac{t}{n}\right) = \lim_{n\to\infty} \ln\left[1+i\mu\frac{t}{n}+o\left(\frac{t}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{i\mu t}{n}. \quad (10)$$

and

$$\lim_{n\to\infty} \ln\varphi_{\frac{S_n}{n}}(t) = \lim_{n\to\infty}\left[\frac{2(i\mu t+2i\mu t+\cdots+mi\mu t)}{n(m+1)}+(n-m)\frac{i\mu t}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2(1+2+\cdots+m)i\mu t}{n(m+1)}+(n-m)\frac{i\mu t}{n}\right]$$

$$= \lim_{n\to\infty}\left(\frac{m}{n}+\frac{n-m}{n}\right)i\mu t = i\mu t. \quad (11)$$

Thus, $\lim_{n\to\infty}\varphi_{\frac{S_n}{n}}(t)=e^{i\mu t}$ or $\frac{S_n}{n}\xrightarrow{P}\mu$ as $n\to\infty$, this is equivalent to

$$\forall\varepsilon>0,\exists n_0,\forall n>n_0: P\left(\left|\frac{S_n}{n}-\mu\right|>\varepsilon\right)<\varepsilon. \quad (12)$$

Since $N_n\to\infty$ as $n\to\infty$, then

$$\lim_{n\to\infty} P(N_n>\delta)=1, \forall\delta>0. \quad (13)$$

In addition, for every $\varepsilon>0$,

$$P\left(\left|\frac{S_{N_n}}{E(N_n)}-\mu\right|>\varepsilon\right)$$

$$=\sum_{k=1}^{\infty} P(N_n=k)P\left(\left|\frac{S_k}{k}-\mu\right|>\varepsilon\right)$$

$$=\sum_{k=1}^{n_0} P(N_n=k)P\left(\left|\frac{S_k}{k}-\mu\right|>\varepsilon\right) \quad (13)$$

$$+\sum_{k=n_0+1}^{\infty} P(N_n=k)P\left(\left|\frac{S_k}{k}-\mu\right|>\varepsilon\right)$$

$$< P(N_n\le n_0)+\varepsilon P(N_n>n_0).$$

By (13), ε is small enough, hence

$$\frac{S_{N_n}}{E(N_n)}\xrightarrow{P}\mu \text{ as } n\to\infty. \quad (14)$$

Finally,

$$\frac{S_{N_n}}{n}\xrightarrow{P}\alpha\mu \text{ as } n\to\infty. \quad (15)$$

2.2 Poisson approximation problem for the extension random sums of *m*-dependent random variables

In this section, we will solve the Poisson approximation problem for the more general case, which is the extension random sums of *m*-dependent random variables, instead of just considering the case of independent random variables (Tran Loc Hung and Le Truong Giang, 2016).

Let S_n be the random variable built as in section 2.1; however, $\{Z_i\}_{i\ge1}$ is Bernoulli distributed random variables with parameter $p_n>0$. Then, $E(S_n)=np_n$ and

$$\varphi_{S_n}(t) = \varphi^2_{Z_1}\left(\frac{t}{(m+1)}\right).\varphi^2_{Z_1}\left(\frac{2t}{(m+1)}\right) ...\varphi^2_{Z_1}\left(\frac{mt}{(m+1)}\right)\left[\varphi_{Z_1}(t)\right]^{n-m} (n=1,2,\dots). \quad (16)$$

Theorem 2.2. Assume that $p_n\to0$ and $np_n\to\lambda$ as $n\to\infty$. Then

$$S_{N_n}\xrightarrow{d}Z_\lambda, \quad (17)$$

where Z_λ is a Poisson distributed random variable with parameter $\lambda>0$.

Proof.

The characteristic function of S_n is given by

$$\varphi_{S_n}(t) = \varphi^2_{Z_1}\left(\frac{t}{(m+1)}\right).\varphi^2_{Z_1}\left(\frac{2t}{(m+1)}\right) ...\varphi^2_{Z_1}\left(\frac{mt}{(m+1)}\right)\left[\varphi_{Z_1}(t)\right]^{n-m}. \quad (18)$$

Take the logarithms and limit of two sides to yield

$$\lim_{n\to\infty} \ln \varphi_{S_n}(t)$$

$$= \lim_{n\to\infty} 2\left[\ln\left(p_n\left(e^{\frac{it}{m+1}}-1\right)+1\right)+\ln\left(p_n\left(e^{\frac{i2t}{m+1}}-1\right)+1\right)+...+\ln\left(p_n\left(e^{\frac{imt}{m+1}}-1\right)+1\right)\right]$$

$$+ \lim_{n\to\infty} (n-m)\ln\left(p_n\left(e^{it}-1\right)+1\right)$$

$$= \lim_{n\to\infty} 2\left[p_n\left(e^{\frac{it}{m+1}}-1\right)+p_n\left(e^{\frac{i2t}{m+1}}-1\right)+...+p_n\left(e^{\frac{imt}{m+1}}-1\right)\right]+\lim_{n\to\infty}(n-m)\,p_n\left(e^{it}-1\right)$$

$$= \lim_{n\to\infty}\left[2p_n\left(e^{\frac{it}{m+1}}+e^{\frac{i2t}{m+1}}+...+e^{\frac{imt}{m+1}}-m\right)+np_n\left(e^{it}-1\right)-mp_n\left(e^{it}-1\right)\right]=\lambda\left(e^{it}-1\right).$$

Hence

$$\lim_{n\to\infty}\varphi_{S_n}(t)=e^{\lambda\left(e^{it}-1\right)} \tag{19}$$

or

$$\forall\varepsilon>0,\exists n_0:\forall n>n_0,\left|P(S_n\le x)-P(Z_\lambda\le x)\right|<\varepsilon. \tag{20}$$

Furthermore,

$$\left|P\left(S_{N_n}\le x\right)-P(Z_\lambda\le x)\right|$$

$$=\left|\sum_{k=1}^{\infty}P(N_n=k)P(S_k\le x)-P(Z_\lambda\le x)\right|$$

$$\le\sum_{k=1}^{\infty}P(N_n=k)\left|P(S_k\le x)-P(Z_\lambda\le x)\right|$$

$$=\sum_{k=1}^{n_0}P(N_n=k)\left|P(S_k\le x)-P(Z_\lambda\le x)\right|$$

$$+\sum_{k=n_0+1}^{\infty}P(N_n=k)\left|P(S_k\le x)-P(Z_\lambda\le x)\right|$$

$$<2P\left(N_n\le n_0\right)+\varepsilon P\left(N_n>n_0\right).$$

Since $N_n\to\infty$ as $n\to\infty$, we have that

$$\lim_{n\to\infty}P(N_n\le\delta)=0,\forall\delta>0.$$

Thus

$$\left|P\left(S_{N_n}\le x\right)-P(Z_\lambda\le x)\right|\to 0\ (n\to\infty). \tag{21}$$

This finishes the proof.

3 CONCLUSIONS

The article has been defined as an asymptotic shape for the extension random sums of the *m*-dependent variables via the weak law of large numbers, and the Poisson approximation problem. The received results are extensions and generalizations of results in Hoffding and Robbins (1948), Dianada (1955), Van der Genugten (1989), Tran Loc Hung and Le Truong Giang (2016), and Le Truong Giang and Tran Loc Hung (2018). It is more interesting when the rates of convergence of the extension random sums of the *m*-dependent random variables are studied. This matter shall be continued studying in future research.

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