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New bounds on poisson approximation for random sums of independent negative-binomial randomvariables

Le Truong Giang [1*] and Trinh Huu Nghiem[2]

[1]*University of Finance - Marketing, Vietnam*

[2]*Nam Can Tho University, Vietnam*

[*]*Correspondence: Le Truong Giang (email: ltgiang@ufm.edu.vn)*

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ABSTRACT

The aim of this paper is to establish new bounds on Poisson approximation for random sums of independent negative-binomial random variables. The bounds showed in current paper are a uniform bound and a non-uniform bound. The received results in this paper are extensions and generalizations of known results.

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1 INTRODUCTION

In recent times, Poisson approximation problem for random sums of discrete random variables has attracted the attention of mathematicians. Several interesting results can be found in Yannaros (1991), Vellaisamy and Upadhye (2009), Kongudomthrap and Chaidee (2012), Teerapabolarn (2013a), Teerapabolarn (2014), Tran Loc Hung and Le Truong Giang (2014), Tran Loc Hung and Le Truong Giang (2016a, 2016b), and Le Truong Giang and Trinh Huu Nghiem (2017).

Let $X_1, X_2, \ldots$ be a sequence of independent negative-binomial random variables with probabilities

$$P(X_i = k) = C^k_{r_i+k-1}(1-p_i)^k p_i^{r_i},$$

where $p_i \in (0,1); r_i = 1,2,\ldots; i = 1,2,\ldots; k = 0,1,\ldots$.

Let $W_n = \sum_{i=1}^{n} X_i$ and U_{λ_n} be a Poisson random variable with mean

$$\lambda_n = E(W_n) = \sum_{i=1}^{n} r_i(1-p_i)p_i^{-1}.$$

In addition, throughout this paper, d_{TV} is denoted a probability distance of total variation, defined by

$$d_{TV}(X,Y) = \sup_A |P(X \in A) - P(Y \in A)|,$$

where $A \subseteq \mathbb{Z}_+$.

A uniform bound and a non-uniform bound for the distance between the distribution functions of W_n and U_{λ_n} were presented in Tran Loc Hung and Le Truong Giang (2016a) as follows:

$$d_{TV}\left(W_n,U_{\lambda_n}\right)\le\sum_{i=1}^{n}\min\left\{\lambda_n^{-1}\left(1-e^{-\lambda_n}\right)r_i(1-p_i)p_i^{-1},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i} \quad (0.1)$$

and

$$\left|P(W_n\le w_0)-P\left(U_{\lambda_n}\le w_0\right)\right|\le\frac{e^{\lambda_n}-1}{\lambda_n}\sum_{i=1}^{n}\min\left\{\frac{r_i(1-p_i)}{(w_0+1)p_i},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i}, \quad (0.2)$$

where $w_0\in\mathbb{Z}_+:=\{0,1,2,\ldots\}$.

Consider the sum $W_N=\sum_{i=1}^{N}X_i$, where N is a non-negative integer valued random variable and independent of the X_i's. The sum is called random sums of independent negative -binomial random variables. Let $U_{\bar{\lambda}}$ be a Poisson random variable with $\bar{\lambda}=E\left(\bar{\lambda}_N\right)$, where $\bar{\lambda}_N=\sum_{i=1}^{N}r_i(1-p_i)$. Teerapabolarn (2014) gave a uniform bound for the distance between the distribution functions of W_N and $U_{\bar{\lambda}}$ as follows:

$$d_{TV}\left(W_N,U_{\bar{\lambda}}\right)\le\min\left\{1,\sqrt{\frac{2}{\bar{\lambda}e}}\right\}E\left|\bar{\lambda}_N-\bar{\lambda}\right| + \min\left\{E\left(\sum_{i=1}^{N}\frac{r_i(1-p_i)^2}{p_i}\right),E\left(\frac{\sum_{i=1}^{N}\frac{r_i(1-p_i)^2}{p_i}}{\sqrt{2\bar{\lambda}_N e}}\right)\right\}. \quad (0.3)$$

In this paper, some of the bounds on Poisson approximation for random sums of independent negative-binomial random variables with mean $\lambda=E(\lambda_N)$, where $\lambda_N=\sum_{i=1}^{N}r_i(1-p_i)p_i^{-1}$, are presented in Section 2.

2 MAIN RESULTS

The following lemma is necessary to prove the main result, which is directly obtained from Barbour *et al.* (1992).

Lemma 2.1. Let U_{λ_N} and U_λ denote a Poisson random variable with mean λ_N and λ, respectively. Then, for $A\subseteq\mathbb{Z}_+$, the total variation distance between the distributions of U_{λ_N} and U_λ satisfies the following inequality:

$$d_{TV}\left(U_{\lambda_N},U_\lambda\right)\le\min\left\{1,\sqrt{\frac{2}{e\lambda}}\right\}E\left|\lambda_N-\lambda\right|. \quad (0.4)$$

The following theorems present non-uniform and uniform bounds for the distance between the distribution functions of W_N and U_λ, which are the expected results.

2.1 A uniform bound on Poisson approximation for random sums of independent negative-binomial random variables

Theorem 2.1. For $A\subseteq\mathbb{Z}_+$,

$$d_{TV}\left(W_N,U_\lambda\right)\le E\left(\sum_{i=1}^{N}\min\left\{\frac{1-e^{-\lambda_N}}{\lambda_N}r_i(1-p_i)p_i^{-1},1-p_i^{r_i}\right\}(1-p_i)p_i^{-1}\right) + \min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E\left|\lambda_N-\lambda\right|. \quad (0.5)$$

Proof. Applying the result in Tran Loc Hung and Le Truong Giang (2016a), the following inequality is satisfied

$$d_{TV}\left(W_n,U_{\lambda_n}\right)\le\sum_{i=1}^{n}\min\left\{\lambda_n^{-1}\left(1-e^{-\lambda_n}\right)r_i(1-p_i)p_i^{-1},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i}. \quad (0.6)$$

From the triangular inequality, combining (1.4) and (1.6), it follows the fact that

$$d_{TV}\left(W_N,U_\lambda\right)=\sum_{n=1}^{\infty}P(N=n)d_{TV}(W_n,U_\lambda)$$

$$\le\sum_{n=1}^{\infty}P(N=n)\left[d_{TV}\left(W_n,U_{\lambda_n}\right)+d_{TV}\left(U_{\lambda_n},U_\lambda\right)\right]$$

$$=\sum_{n=1}^{\infty}P(N=n)d_{TV}\left(W_n,U_{\lambda_n}\right)+d_{TV}\left(U_{\lambda_N},U_\lambda\right)$$

$$\le\sum_{n=1}^{\infty}P(N=n)\sum_{i=1}^{n}\min\left\{\frac{\left(1-e^{-\lambda_n}\right)r_i(1-p_i)}{\lambda_n p_i},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i}$$

$$+\min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E|\lambda_N-\lambda|$$

$$\le E\left(\sum_{i=1}^{N}\min\left\{\lambda_N^{-1}\left(1-e^{-\lambda_N}\right)r_i(1-p_i)p_i^{-1},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i}\right)$$

$$+\min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E\left|\lambda_N-\lambda\right|.$$

This finishes the proof.

Remark 2.1. The result of (1.5) is interesting because of considering $\lambda_N=\sum_{i=1}^{N}r_i(1-p_i)p_i^{-1}$ instead of $\bar{\lambda}_N=\sum_{i=1}^{N}r_i(1-p_i)$ as in Teerapabolarn (2014). It is easily seen that the (1.1) is a special case of the (1.5) when $N=n\in\mathbb{Z}_+$ is fixed.

Corollary 2.1. For $r_1=r_2=...=r_n=1$, then

$$d_{TV}\left(W_N,U_\lambda\right)\le E\left(\sum_{i=1}^{N}\min\left\{\lambda_N^{-1}\left(1-e^{-\lambda_N}\right)p_i^{-1},1\right\}(1-p_i)^2p_i^{-1}\right) \quad (0.7)$$
$$+\min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E\left|\lambda_N-\lambda\right|.$$

Remark 2.2. The result (1.7) is a Poisson approximation for the random sums of independent geometric random variables, which is introduced in Teerapabolarn (2013a).

2.2 A non-uniform bound on Poisson approximation for random sums of independent negative-binomial random variables

Theorem 2.2. For $w_0\in\mathbb{Z}_+$, we have

$$\left|P(W_N\le w_0)-P(U_\lambda\le w_0)\right|\le\min\left\{\frac{2\lambda}{w_0+1},\min\left\{1,\sqrt{\frac{2}{e\lambda}}\right\}E|\lambda_N-\lambda|\right\}$$
$$+E\left(\lambda_N^{-1}\left(e^{\lambda_N}-1\right)\sum_{i=1}^{N}\min\left\{\frac{r_i(1-p_i)}{p_i(w_0+1)},1-p_i^{r_i}\right\}(1-p_i)p_i^{-1}\right). \quad (0.8)$$

Proof. Applying the corresponding results in Tran Loc Hung and Le Truong Giang (2016a) and Teerapabolarn (2013a) yields

$$\left|P(W_n\le w_0)-\sum_{k\le w_0}\frac{\lambda_n^k e^{-\lambda_n}}{k!}\right|\le\frac{e^{\lambda_n}-1}{\lambda_n}\sum_{i=1}^{n}\min\left\{\frac{r_i(1-p_i)}{p_i(w_0+1)},1-p_i^{r_i}\right\}\frac{1-p_i}{p_i} \quad (0.9)$$

and

$$\left|P\left(U_{\lambda_N}\le w_0\right)-P(U_\lambda\le w_0)\right|\le\min\left\{\frac{2\lambda}{w_0+1},\min\left\{1,\sqrt{\frac{2}{e\lambda}}\right\}E|\lambda_N-\lambda|\right\}. \quad (0.10)$$

Combining (1.9) and (1.10) gives

$$\left|P(W_N \le w_0) - P(U_\lambda \le w_0)\right| \le \sum_{n=0}^{\infty} P(N=n)\left|P(W_n \le w_0) - P(U_\lambda \le w_0)\right|$$

$$\le \sum_{n=0}^{\infty} P(N=n)\left[\left|P(W_n \le w_0) - P\left(U_{\lambda_n} \le w_0\right)\right| + \left|P\left(U_{\lambda_n} \le w_0\right) - P(U_\lambda \le w_0)\right|\right]$$

$$\le \sum_{n=0}^{\infty} P(N=n)\left|P(W_n \le w_0) - P\left(U_{\lambda_n} \le w_0\right)\right|$$

$$+\left|P\left(U_{\lambda_N} \le w_0\right) - P(U_\lambda \le w_0)\right|$$

$$\le \sum_{n=0}^{\infty} P(N=n)\frac{e^{\lambda_n}-1}{\lambda_n}\sum_{i=1}^{n}\min\left\{\frac{r_i(1-p_i)}{p_i(w_0+1)}, 1-p_i^{r_i}\right\}\frac{1-p_i}{p_i}$$

$$+\min\left\{\frac{2\lambda}{w_0+1}, \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\}E\left|\lambda_N - \lambda\right|\right\}$$

$$\le E\left(\lambda_N^{-1}\left(e^{\lambda_N}-1\right)\sum_{i=1}^{N}\min\left\{\frac{r_i(1-p_i)}{p_i(w_0+1)}, 1-p_i^{r_i}\right\}(1-p_i)p_i^{-1}\right)$$

$$+\min\left\{\frac{2\lambda}{w_0+1}, \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\}E\left|\lambda_N - \lambda\right|\right\}.$$

The proof is completed.

Remark 2.3. It is easily to check that the (1.2) is a special case of the (1.8) when $N=n\in\mathbb{Z}_+$ is fixed.

Corollary 2.2. For $r_1=r_2=...=r_n=1$, then

$$\left|P(W_N \le w_0) - P(U_\lambda \le w_0)\right| \le \min\left\{\frac{2\lambda}{w_0+1}, \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\}E\left|\lambda_N - \lambda\right|\right\}$$
$$+E\left(\lambda_N^{-1}\left(e^{\lambda_N}-1\right)\sum_{i=1}^{N}\min\left\{\frac{1}{p_i(w_0+1)}, 1\right\}\frac{(1-p_i)^2}{p_i}\right). \quad (0.11)$$

Remark 2.4. The result (1.11) is a non-uniform bound on Poisson approximation for the random sums of independent geometric random variables.

3 CONCLUSIONS

Bounds for the distance between the distribution function of random sums of independent negative-binomial random variables and an appropriate Poisson distribution function were obtained. The results in this paper are extensions and generalizations of results in Teerapabolarn (2013a), and Teerapabolarn (2014), Tran Loc Hung and Le Truong Giang (2016a, 2016b). The results will be more interesting and valuable if Poisson approximation for random sums of dependent negative - binomial random variables is discussed.

REFERENCES

Barbour, A. D., Holst, L. and Janson, S., 1992. Poisson Approximation, Clarendon Press-Oxford, 277.

Kongudomthrap, S. and Chaidee, N., 2012. Bounds in Poisson approximation of random sums of Bernoulli random variables. Journal of Mathematics Research. 4(3): 29-35.

Le Truong Giang and Trinh Huu Nghiem, 2017. Characteristic functions method for some of the limit theorems in probability. Can Tho University Journal of Science. 53: 88-95.

Teerapabolarn, K., 2013a. Improved bounds on Poisson approximation for independent binomial random summands. International Journal of Pure and Applied Mathematics. 89(1): 29-33.

Teerapabolarn, K., 2013b. Poisson approximation for random sums of geometric random variables. International Journal of Pure and Applied Mathematics. 89(1): 35-39.

Teerapabolarn, K., 2014. Poisson approximation for random sums of independent negative binomial random variables. International Journal of Pure and Applied Mathematics. 93(6): 783-787.

Tran Loc Hung and Le Truong Giang, 2014. On bounds in Poisson approximation for integer-valued independent random variables. Journal of Inequalities and Applications, https://doi.org/10.1186/1029-242X-2014-291.

Tran Loc Hung and Le Truong Giang, 2016a. On bounds in Poisson approximation for distributions of independent negative-binomial distributed random variables. SpringerPlus, https://doi.org/10.1186/s40064-016-1710-y.

Tran Loc Hung and Le Truong Giang, 2016b. On the bounds in Poisson approximation for independent geometric distributed random variables. Bulletin of the Irannian Mathematical Society. 42(5): 1087-1096.

Vellaisamy, P. and Upadhye, S., 2009. Compound negative binomial approximations for sums of random variables. Probability and Mathematical Statistics, 29(2): 205 - 226.

Yannaros, N., 1991. Poisson approximation for random sums of Bernoulli random variables. Statistics & Probability Letters. 11(2): 161-165.